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Existence results for an evolution problem with fractional nonlocal conditions

Nasser-eddine Tatar *

King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran 31261, Saudi Arabia

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ABSTRACT

A second-order abstract problem involving derivatives of non-integer order in the nonlinearity as well as in the nonlocal conditions is considered. This problem covers many situations arising in real world phenomena and extends the case of integer order to the case of non-integer order. We establish the existence of mild solutions. Under extra appropriate sufficient assumptions on the nonlinearity and on the initial data these mild solutions are shown to be classical.

1. Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1–6] and references therein (to cite but a few) for the case of abstract second-order differential equations. In contrast, one cannot find papers in the literature dealing with nonlocal conditions involving fractional derivatives. Nevertheless, we may find a few papers treating well-posedness and asymptotic behavior of solutions for some problems with boundary conditions containing fractional derivatives (see [7–12]). In [13] the present author has introduced nonlocal conditions involving fractional derivatives and called them "nonlocal conditions of fractional type". The order of these fractional derivatives as well as the ones in the nonlinearity were between 0 and 1. In the present paper we consider the case where the orders are between 1 and 2. Therefore, this paper may be regarded as a continuation of the paper [13]. In the integer order case, the underlying space in which to look for mild solutions is the space of continuously differentiable functions. If the derivatives are of order between 1 and 2, more regularity is needed for the initial data and also for the nonlinearity. There are also several other differences from the already treated case where the derivatives were between 0 and 1. In particular, the continuity of the fractional derivative of a (continuous) function does not imply the continuity of its fractional derivatives of lower orders.

The problem of concern is the following general Cauchy problem:

$$\begin{cases} u''(t) = Au(t) + f\left(t, u(t), D^{\alpha_1}u(t), \dots, D^{\alpha_n}u(t)\right) \\ \quad + \displaystyle\int_0^t g\left(s, u(s), D^{\nu_1}u\left(s\right), \dots, D^{\nu_n}u(s)\right) \mathrm{d}s, \quad t \in I = [0, T] \\ u(0) = u^0 + p\left(u, D^{\beta_1}u(t), \dots, D^{\beta_m}u(t)\right), \\ u'(0) = u^1 + q\left(u, D^{\gamma_1}u(t), \dots, D^{\gamma_r}u(t)\right) \end{cases}$$

* Tel.: +966 3 860 4654; fax: +966 3 860 2340.
E-mail address: tatarn@kfupm.edu.sa.

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where $0 < \alpha_i \le 1$ and $0 < \beta_j, \gamma_k, \nu_l < 2$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, r$, $l = 1, \ldots, z$. Here the prime denotes time differentiation and D^{ρ_l}, $\rho_\delta = \alpha_\delta, \beta_\delta, \gamma_\delta, \nu_\delta$, $\delta = 1, \ldots, n$, $\delta = 1, \ldots, m$, $\delta = 1, \ldots, r$, $\delta = 1, \ldots, z$, respectively, denotes fractional time differentiation (in the Riemann–Liouville sense). The operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \ge 0$, of bounded linear operators in the Banach space X and f, g are nonlinear functions from $\mathbf{R}^+ \times X \times \cdots \times X$ to X, and u^0 and u^1 are given initial data in X. The functions $p : [C(I; X)]^{m+1} \to X$, $q : [C(I; X)]^{r+1} \to X$ are given continuous functions. This problem has been studied in the cases where $\alpha_i, \beta_j, \gamma_k, \nu_l$ are 0 or 1 (see [1–3,14,5,6,15]). Well-posedness has been proved using different methods such as ones based on fixed point theorems and the theory of strongly continuous cosine families in Banach spaces. We refer the reader to [16–18] for a good account on the theory of the cosine family. The case $p = q = g \equiv 0$, $0 < \alpha_i \le 2$, $i = 1, \ldots, n$, has been investigated by the present author (with Kirane and Medved') in [19]. Several results on classical solutions and mild solutions have been proved under different conditions on the nonlinearities and the initial data.

Here we consider the case where $0 < \alpha_i < 1$, $i = 1, \ldots, n$, and $0 < \beta_j, \gamma_k, \nu_l < 2$ $j = 1, \ldots, m$, $k = 1, \ldots, r$, $l = 1, \ldots, z$. That is, we consider nonlocal conditions involving fractional derivatives which are the natural generalization of the nonlocal conditions of integer order type. Moreover, the Lipschitz continuity of the partial derivatives of the nonlinearity f is dropped. This contribution may be regarded as a generalization of the work of Hernandez [5] and an extension of [13].

To lighten the exposition of this work we will treat the problem

$$\begin{cases} u''(t) = Au(t) + f\left(t, u(t), t^{\lambda_\alpha} D^\alpha u(t)\right) \\ \quad + \displaystyle\int_0^t g\left(s, u(s), t^{\lambda_\nu} D^\nu u(s)\right) ds, \quad t \in I = [0, T] \\ u(0) = u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right), \\ u'(0) = u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right) \end{cases} \tag{1}$$

with $0 < \alpha < 1$ and $0 < \beta, \gamma, \nu < 2$, $\lambda_\eta \ge 0$, $\eta = \alpha, \beta, \gamma, \nu$. To make the problem more interesting we suppose further that at least one of β, γ or ν is between 1 and 2.

The next section of this paper contains some notation and preliminary results needed in our proofs. Section 3 treats the existence and uniqueness of a mild solution in an appropriate space. Section 4 is devoted to the existence and uniqueness of a classical solution.

2. Preliminaries

In this section we present some notation, assumptions and results needed in our proofs later.

Definition 1. The integral

$$(I^\alpha_{a+} h)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h(t)dt}{(x-t)^{1-\alpha}}, \quad x > a$$

is called the Riemann–Liouville fractional integral of h of order $\alpha > 0$ when the right side exists.

Here Γ is the usual Gamma function

$$\Gamma(z) := \int_0^\infty e^{-s} s^{z-1} ds, \quad z > 0.$$

Definition 2. The (left hand) Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined by

$$(D^\alpha_a h)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{h(t)dt}{(x-t)^{\alpha-n+1}}, \quad x > a, \ n = [\alpha] + 1$$

whenever the right side is pointwise defined.

In particular

$$(D^\gamma_a h)(x) = \frac{1}{\Gamma(2-\gamma)} \left(\frac{d}{dx}\right)^2 \int_a^x \frac{h(t)dt}{(x-t)^{\gamma-1}}, \quad x > a, \ 1 < \gamma < 2.$$

See [20–24] for more on fractional derivatives and fractional integrals.

We will assume that

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbf{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t)$, $t \in \mathbf{R}$, is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad t \in \mathbf{R}, \ x \in X. \tag{2}$$

It is known (see [21,22,25]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq M e^{\omega|t|}, \quad t \in \mathbf{R} \quad \text{and} \quad |S(t) - S(t_0)| \leq M \left| \int_{t_0}^{t} e^{\omega|s|} ds \right|, \quad t, t_0 \in \mathbf{R}.$$

For simplicity we will write $|C(t)| \leq \tilde{M}$ and $|S(t)| \leq \tilde{N}$ on $I = [0, T]$ (of course $\tilde{M} \geq 1$ and $\tilde{N} \geq 1$ depend on T).

If we define

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbf{R}\} \tag{3}$$

then we have:

Lemma 1 (*See [17,18,15]*). *Assume that* (H1) *is satisfied. Then*

(i) $S(t)X \subset E$, $t \in \mathbf{R}$,
(ii) $S(t)E \subset D(A)$, $t \in \mathbf{R}$,
(iii) $\frac{d}{dt} C(t)x = AS(t)x$, $x \in E$, $t \in \mathbf{R}$,
(iv) $\frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, $t \in \mathbf{R}$.

Lemma 2 (*See [17,18,15]*). *Suppose that* (H1) *holds, with $v : \mathbf{R} \to X$ a continuously differentiable function and $q(t) = \int_0^t S(t-s)v(s)ds$. Then, $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ and $q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t)$.*

Definition 3. A continuous solution such that $D^\eta u \in C(I, X)$, $\eta = \alpha, \beta, \gamma, \nu$, of the integro-fractional-differential equation

$$\begin{aligned} u(t) &= C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] + S(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right)\right] \\ &\quad + \int_0^t S(t-s)\left[f\left(s, u(s), t^{\lambda_\alpha} D^\alpha u(s)\right) + \int_0^s g\left(z, u(z), t^{\lambda_\nu} D^\nu u(z)\right) dz\right] ds \end{aligned} \tag{4}$$

for $t \in I$ is called a mild solution of problem (1).

The following lemmas will be very useful later. The first two can be found in [24].

Lemma 3. *If $\varphi(x) \in AC^n([a, b])$ where*

$$AC^n([a, b]) := \left\{\phi : [a, b] \to \mathbf{R} \text{ and } (D^{n-1}\phi)(x) \in AC[a, b]\right\},$$

$\alpha > 0$ and $n = [\alpha] + 1$, then

$$(D_a^\alpha \varphi)(x) = \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\varphi^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, \quad x > a.$$

Lemma 4. *Let $\alpha > 0$, $\beta < 0$ and $\varphi \in L^1(a, b)$ be such that $I^{n+\beta}\varphi \in AC^n([a, b])$. Then*

$$I_{a+}^\alpha I_{a+}^\beta \varphi = I_{a+}^{\alpha+\beta}\varphi - \sum_{k=0}^{n-1} \frac{\varphi_{n+\beta}^{(n-k-1)}(a)}{\Gamma(\alpha-k)}(x-a)^{\alpha-k-1}$$

where $\varphi_{n+\beta}(x) = I_{a+}^{n+\beta}\varphi(x)$ and $n = [-\beta] + 1$.

Lemma 5 (*[20], Property 2.4*). *Let $\alpha > 0$, $\beta > 0$ be such that $n-1 < \alpha \leq n$, $m-1 < \beta \leq m$ ($n, m \in \mathbf{N}$) and $\alpha + \beta < n$, and let $\varphi \in L^1(a, b)$ and $\varphi_{m-\alpha} \in AC^m([a, b])$. Then we have the following index rule:*

$$\left(D_{a+}^\alpha D_{a+}^\beta \varphi\right)(x) = \left(D_{a+}^{\alpha+\beta}\varphi\right)(x) - \sum_{j=1}^{m} \frac{\left(D_{a+}^{\beta-j}\varphi\right)(a+)}{\Gamma(1-j-\alpha)}(x-a)^{-j-\alpha}.$$

Lemma 6 (*[20, Property 2.2 and Lemma 2.9(c)]*). *If $\alpha > \beta > 0$, then the formula*

$$\left(D_{a+}^\beta I_{a+}^\alpha \varphi\right)(x) = \left(I_{a+}^{\alpha-\beta}\varphi\right)(x)$$

holds almost everywhere on $[a, b]$ for $\varphi \in L^1(a, b)$ and it is valid at any point $x \in [a, b]$ in the case where $\varphi \in C([a, b])$.

3. Existence of mild solutions

In this section we prove existence and uniqueness of a mild solution in the space

$$C_\eta^{RL}(I; X) := \{v \in C(I; X) : t^\eta D^\eta v \in C(I; X),\ \eta = \alpha, \beta, \gamma, \nu\}$$

equipped with the norm $\|v\|_\eta := \|v\|_C + \sum_\eta \|t^\eta D^\eta v\|_C$ where $\|.\|_C$ is the sup norm in $C(I;X)$. We also define the space

$$E_\eta := \{x \in X : t^\eta D^\eta C(t)x \text{ is continuous on } \mathbf{R}^+,\ \eta = \alpha, \beta, \gamma, \nu\}.$$

The assumptions on f, g, p and q are:
(H2) f, $t^{\eta_f} D^{\eta-1} f$ $(0 < \eta_f < 1)$, $g : \mathbf{R}^+ \times X \times X \to X$ are continuous and satisfy the Lipschitz conditions

$$\begin{aligned}
&\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le A_f \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right),\\
&\left\|t^{\eta_f} D^{\eta-1} f(t,x_1,y_1) - t^{\eta_f} D^{\eta-1} f(t,x_2,y_2)\right\| \le \tilde{A}_{\eta_f} \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right),\\
&\|g(t,x_1,y_1) - g(t,x_2,y_2)\| \le A_g \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right)
\end{aligned}$$

for $x_1, y_1, x_2, y_2 \in X$, $t \in I$ and some positive constants A_f, $\tilde{A}_{\eta_f}$ and A_g.
(H3) $p : [C(I;X)]^2 \to E_\eta$ and $q : [C(I;X)]^2 \to X$ are continuous and satisfy

$$\begin{aligned}
&\|p(x_1,y_1) - p(x_2,y_2)\| \le A_p \left(\|x_1 - x_2\|_C + \|y_1 - y_2\|_C\right),\\
&\|q(x_1,y_1) - q(x_2,y_2)\| \le A_q \left(\|x_1 - x_2\|_C + \|y_1 - y_2\|_C\right),
\end{aligned}$$

for $x_1, y_1, x_2, y_2 \in C(I;X)$ and some positive constants A_p and A_q.

Lemma 7 ([4]). *Let $\alpha > 0$ and $\beta > 0$ be such that $n - 1 < \alpha \le n$, $m - 1 < \beta \le m$ $(n, m \in \mathbf{N})$. If $D^\beta_{a+} f$ exists and is finite on $[a,b]$ and is such that $D^\alpha_{a+}(D^\beta_{a+} f)$ exists also and is finite, then*

$$D^\alpha_{a+} D^\beta_{a+} f(x) = D^{\alpha+\beta}_{a+} f(x) - \sum_{k=1}^{m} \frac{A_k}{\Gamma(1-\alpha-k)} (x-a)^{-\alpha-k}, \quad x \in [a,b]$$

where $A_k = \lim_{x\to a+} D^{m-k}_{a+} I^{m-\beta}_{a+} f(x)$, $k = 1, 2, \ldots, m$, provided that

(i) $n + m - \alpha - \beta \ge 1$ *or*
(ii) $n + m - \alpha - \beta < 1$ *and f is such that $|x-a|^\lambda f(x)$ is continuous on $[a,b]$ for some $\lambda \in [0, 1-\gamma)$, with $1 - n - m + \alpha + \beta \le \gamma \le 1$.*

The next lemma is proved in [13] and is reported here with its proof just for completeness.

Lemma 8. *If $R(t)$ is a linear operator such that $I^{1-\nu} R(t)x \in C^1([0,T])$, $T > 0$, then, for $0 < \nu < 1$, we have*

$$D^\nu \int_0^t R(t-s)x\mathrm{d}s = \int_0^t D^\nu R(t-s)x\mathrm{d}s + \lim_{t\to 0^+} I^{1-\nu} R(t)x, \quad x \in X,\ t \in [0,T].$$

Proof. By Definition 2 and the assumption $I^{1-\nu}R(t)x \in C^1([0,T])$, we have

$$\begin{aligned}
D^\nu \int_0^t R(t-s)x\mathrm{d}s &= \frac{1}{\Gamma(1-\nu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\mathrm{d}\tau}{(t-\tau)^\nu} \int_0^\tau R(\tau-s)x\mathrm{d}s\\
&= \frac{1}{\Gamma(1-\nu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{d}s \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu} \mathrm{d}\tau\\
&= \frac{1}{\Gamma(1-\nu)} \int_0^t \mathrm{d}s \frac{\partial}{\partial t} \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu} \mathrm{d}\tau + \frac{1}{\Gamma(1-\nu)} \lim_{s\to t^-} \int_s^t \frac{R(\tau-s)x}{(t-\tau)^\nu} \mathrm{d}\tau, \quad t \in [0,T].
\end{aligned}$$

Moreover, a change of variable $\sigma = \tau - s$ leads to

$$D^\nu \int_0^t R(t-s)x\mathrm{d}s = \int_0^t \frac{\mathrm{d}s}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^{t-s} \frac{R(\sigma)x}{(t-s-\sigma)^\nu} \mathrm{d}\sigma + \frac{1}{\Gamma(1-\nu)} \lim_{t\to 0^+} \int_0^t \frac{R(\sigma)x}{(t-\sigma)^\nu} \mathrm{d}\sigma$$

and the proof is complete. □

Now we prove a similar result for $1 < \nu < 2$.

Lemma 9. *Let $R(t)$ be a continuously differentiable linear operator on I and h continuous (with values in X) be such that $I^{2-\nu} h \in C^1([0,T])$. Then, for $1 < \nu < 2$, we have*

$$D^\nu \int_0^t R(t-s)h(s)\mathrm{d}s = \int_0^t R'(s) D^{\nu-1} h(t-s)\mathrm{d}s + \left(\lim_{t\to 0^+} R(t)\right) D^{\nu-1} h(t).$$

Proof. By the assumption $I^{2-\nu}h \in C^1([0,T])$ we infer that $D^{\nu-1}h(s)$ exists and is continuous on I. Moreover, Lemma 7 or formula (2.122) in [23]

$$D^{\rho}\left(\frac{\mathrm{d}^n k(t)}{\mathrm{d}t^n}\right) = D^{\rho+n}k(t) - \sum_{j=0}^{n-1}\frac{k^{(j)}(0)t^{j-\rho-n}}{\Gamma(1+j-\rho-n)}$$

gives us

$$\begin{aligned} D^{\nu}\int_0^t R(t-s)h(s)\mathrm{d}s &= D^{\nu-1+1}\int_0^t R(t-s)h(s)\mathrm{d}s \\ &= D^{\nu-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t R(t-s)h(s)\mathrm{d}s = D^{\nu-1}\left[\int_0^t R'(t-s)h(s)\mathrm{d}s + \left(\lim_{t\to 0^+}R(t)\right)h(t)\right] \\ &= D^{\nu-1}\left[\int_0^t R'(s)h(t-s)\mathrm{d}s + \left(\lim_{t\to 0^+}R(t)\right)h(t)\right]. \end{aligned}$$

Our assumption that $I^{2-\nu}h \in C^1([0,T])$ allows us to apply the proof of the previous lemma to obtain

$$D^{\nu}\int_0^t R(t-s)h(s)\mathrm{d}s = \int_0^t R'(s)D^{\nu-1}h(t-s)\mathrm{d}s + \left(\lim_{t\to 0^+}R(t)\right)D^{\nu-1}h(t) + R'(t)\lim_{t\to 0^+}I^{2-\nu}h(t).$$

This completes the proof. □

Corollary 1. *Let $S(t)$ be the associated sine family with the cosine family $C(t)$, with $1<\nu<2$ and let h be a continuous function such that $I^{2-\nu}h \in C^1([0,T])$. Then, we have for $t\in[0,T]$*

$$D^{\nu}\int_0^t S(t-s)h(s)\mathrm{d}s = \int_0^t C(s)D^{\nu-1}h(t-s)\mathrm{d}s.$$

Proof. This is a consequence of the fact that $S(t)$ is continuously differentiable, the previous lemma and

$$\begin{aligned} D^{\nu}\int_0^t S(t-s)h(s)\mathrm{d}s &= \int_0^t C(s)D^{\nu-1}h(t-s)\mathrm{d}s + \left(\lim_{t\to 0^+}S(t)\right)D^{\nu-1}h(t) \\ &= \int_0^t C(s)D^{\nu-1}h(t-s)\mathrm{d}s. \quad \square \end{aligned}$$

We are now ready to state and prove our first main result. Just to fix ideas we will assume in the rest of the paper that $0<\alpha<1$ and $1<\beta,\gamma,\nu<2$.

Theorem 1. *Assume that* (H1)–(H3) *hold and $\lambda_\eta \ge \eta$, $\eta=\alpha,\beta,\gamma,\nu$. Let $u^0+p(u,D^{\beta}u(t)) \in E_\eta$ and $u^1+q\left(u,D^{\gamma}u(t)\right) \in E_{\eta-1}$. If $\tilde{M},\tilde{N},R$ (a bound for $t^{\eta}D^{\eta}C(t)$ on I) and $\tilde{R}$ (a bound for $t^{\eta}D^{\eta-1}C(t)$ on I) and T are sufficiently small, then problem (1) admits a unique mild solution $u\in C_\eta^{RL}([0,T])$.*

Proof. Let us define the function

$$\begin{aligned} \Phi(u)(t) := {}& C(t)\left[u^0+p(u,t^{\lambda_\beta}D^{\beta}u(t))\right] + S(t)\left[u^1+q\left(u,t^{\lambda_\gamma}D^{\gamma}u(t)\right)\right] \\ &+ \int_0^t S(t-s)f\left(s,u(s),s^{\lambda_\alpha}D^{\alpha}u(s)\right)\mathrm{d}s + \int_0^t S(t-s)\int_0^s g\left(z,u(z),z^{\lambda_\nu}D^{\nu}u(z)\right)\mathrm{d}z\mathrm{d}s \end{aligned} \tag{5}$$

and take its η-th ($\eta=\beta,\gamma,\nu$) fractional derivative (using Lemma 7)

$$\begin{aligned} t^{\eta}D^{\eta}\Phi(u)(t) = {}& t^{\eta}D^{\eta}C(t)\left[u^0+p(u,t^{\lambda_\beta}D^{\beta}u(t))\right] + t^{\eta}D^{\eta-1}C(t)\left[u^1+q\left(u,t^{\lambda_\gamma}D^{\gamma}u(t)\right)\right] \\ &+ t^{\eta}D^{\eta}\int_0^t S(t-s)f\left(s,u(s),s^{\lambda_\alpha}D^{\alpha}u(s)\right)\mathrm{d}s \\ &+ t^{\eta}D^{\eta}\int_0^t S(t-s)\int_0^s g\left(z,u(z),z^{\lambda_\nu}D^{\nu}u(z)\right)\mathrm{d}s. \end{aligned} \tag{6}$$

Clearly, by our assumptions and for $u\in C_\eta^{RL}([0,T])$, the expressions in (5) and (6) are well-defined. In fact, by Corollary 1, we can rewrite (6) in the following manner:

$$t^{\eta}D^{\eta}\Phi(u)(t) = t^{\eta}D^{\eta}C(t)\left[u^0+p(u,t^{\lambda_\beta}D^{\beta}u(t))\right] + t^{\eta}D^{\eta-1}C(t)\left[u^1+q\left(u,t^{\lambda_\gamma}D^{\gamma}u(t)\right)\right]$$

$$+ t^{\eta}\int_0^t C(t-s)D^{\eta-1}f\left(s,u(s),s^{\lambda_\alpha}D^{\alpha}u(s)\right)\mathrm{d}s$$
$$+ t^{\eta}\int_0^t C(t-s)D^{\eta-1}\int_0^s g\left(z,u(z),z^{\lambda_\nu}D^{\nu}u(z)\right)\mathrm{d}s. \tag{7}$$

Therefore, $\Phi, t^{\eta}D^{\eta}\Phi : C_{\kappa}^{RL}([0,T]) \to C([0,T])$ (see Lemma 6). Next, by assumptions (H2), (H3) and (5), for $u, v \in C_{\eta}^{RL}([0,T])$, we have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\| \le\ & \tilde{M}A_p\left(\|u-v\|_C + \left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C\right) + \tilde{N}A_q\left(\|u-v\|_C + \left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C\right)\\ &+ \tilde{N}A_f\int_0^t\left(\|u-v\| + \left\|s^{\lambda_\alpha}D^{\alpha}u - s^{\lambda_\alpha}D^{\alpha}v\right\|\right)\mathrm{d}s\\ &+ \tilde{N}A_g\int_0^t s\sup_{0\le z\le s}\left(\|u-v\| + \left\|z^{\lambda_\nu}D^{\nu}u - z^{\lambda_\nu}D^{\nu}v\right\|\right)\mathrm{d}s.\end{aligned} \tag{8}$$

Therefore

$$\begin{aligned}\|\Phi(u)-\Phi(v)\| \le\ & \left(\tilde{M}A_p + \tilde{N}A_q\right)\|u-v\|_C + \tilde{M}A_p\left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C\\ &+ \tilde{N}A_q\left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C + \tilde{N}A_f T\sup_{0\le s\le T}\|u-v\|\\ &+ \tilde{N}A_f\int_0^t\left\|s^{\lambda_\alpha}D^{\alpha}u - s^{\lambda_\alpha}D^{\alpha}v\right\|\mathrm{d}s + \tilde{N}A_g T^2\sup_{0\le z\le T}\|u-v\|\\ &+ \tilde{N}A_g T\int_0^t\left\|s^{\lambda_\nu}D^{\nu}u - s^{\lambda_\nu}D^{\nu}v\right\|\mathrm{d}s\end{aligned}$$

or

$$\|\Phi(u)-\Phi(v)\| \le C_1\|u-v\|_C + C_2\sum_{\eta}\left\|t^{\lambda_\eta}D^{\eta}u - t^{\lambda_\eta}D^{\eta}v\right\|_C \tag{9}$$

where

$$C_1 = \tilde{M}A_p + \tilde{N}A_q + \tilde{N}A_f T + \tilde{N}A_g T^2$$

and

$$C_2 = \max\left\{\tilde{M}A_p, \tilde{N}A_q, \tilde{N}A_f T, \tilde{N}A_g T^2\right\}.$$

Moreover, by virtue of Lemma 6, we have

$$\begin{aligned}&\left\|\int_0^t C(t-s)D^{\eta-1}\int_0^s\left[g\left(z,u(z),z^{\lambda_\nu}D^{\nu}u(z)\right) - g\left(z,v(z),z^{\lambda_\nu}D^{\nu}v\,(z)\right)\right]\mathrm{d}z\mathrm{d}s\right\|\\ &\quad= \left\|\int_0^t C(t-s)I^{2-\eta}\left[g\left(s,u\,(s)\,,s^{\lambda_\nu}D^{\nu}u(s)\right) - g\left(s,v(s),s^{\lambda_\nu}D^{\nu}v(s)\right)\right]\mathrm{d}s\right\|\\ &\quad\le \tilde{M}A_g\sup_{0\le z\le T}\left(\|u-v\| + \left\|z^{\lambda_\nu}D^{\nu}u - z^{\lambda_\nu}D^{\nu}v\right\|\right)\int_0^t\frac{s^{2-\eta}}{\Gamma(3-\eta)}\mathrm{d}s\\ &\quad\le \frac{\tilde{M}A_g T^{3-\eta}}{\Gamma(4-\eta)}\left(\|u-v\|_C + \left\|t^{\lambda_\nu}D^{\nu}u - t^{\lambda_\nu}D^{\nu}v\right\|_C\right)\end{aligned}$$

and then

$$\begin{aligned}\|t^{\eta}D^{\eta}\Phi(u) - t^{\eta}D^{\eta}\Phi(v)\| \le\ & RA_p\left(\|u-v\|_C + \left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C\right) + \tilde{R}A_q\left(\|u-v\|_C + \left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C\right)\\ &+ \frac{\tilde{M}\tilde{A}_{\eta_f}T^{1-\eta_f+\eta}}{1-\eta_f}\left(\|u-v\|_C + \left\|t^{\lambda_\alpha}D^{\alpha}u - t^{\lambda_\alpha}D^{\alpha}v\right\|_C\right)\\ &+ \frac{\tilde{M}A_g T^3}{\Gamma(4-\eta)}\left(\|u-v\|_C + \left\|t^{\lambda_\nu}D^{\nu}u - t^{\lambda_\nu}D^{\nu}v\right\|_C\right).\end{aligned}$$

Hence,

$$\|t^{\eta}D^{\eta}\Phi(u) - t^{\eta}D^{\eta}\Phi(v)\| \le C_3\|u-v\|_C + C_4\sum_{\eta}\left\|t^{\lambda_\eta}D^{\eta}u - t^{\lambda_\eta}D^{\eta}v\right\|_C \tag{10}$$

where

$$C_3 = RA_p + \tilde{R}A_q + \frac{\tilde{M}\tilde{A}_{\eta_f} T^{1-\eta_f+\eta}}{1-\eta_f} + \frac{\tilde{M}A_g T^3}{\Gamma(4-\eta)}$$

and

$$C_4 = \max\left\{RA_p, \tilde{R}A_q, \frac{\tilde{M}\tilde{A}_{\eta_f} T^{1-\eta_f+\eta}}{1-\eta_f}, \frac{\tilde{M}A_g T^3}{\Gamma(4-\eta)}\right\}.$$

For $\eta = \alpha$ we have by Lemma 8

$$t^\alpha D^\alpha \Phi(u) = t^\alpha D^\alpha C(t)u(0) + t^\alpha D^\alpha S(t)u'(0) + t^\alpha \int_0^t D^\alpha S(t-s) f\left(s, u(s), s^{\lambda_\alpha} D^\alpha u(s)\right) ds$$
$$+ t^\alpha \int_0^t D^\alpha S(t-s) \int_0^s g\left(z, u(z), z^{\lambda_\nu} D^\nu u(z)\right) ds.$$

As $0 < \alpha < 1$, it is clear that

$$\left\| \int_0^t D^\alpha S(t-s) f\left(s, u(s), s^{\lambda_\alpha} D^\alpha u(s)\right) ds \right\| \le \sup_{s\in[0,T]} \|s^\alpha D^\alpha S(s)\| \int_0^t (t-s)^{-\alpha} \left\| f\left(s, u(s), s^{\lambda_\alpha} D^\alpha u(s)\right)\right\| ds$$
$$\le \|t^\alpha D^\alpha S(t)\|_C\, I^{1-\alpha} \left\| f\left(s, u(s), s^{\lambda_\alpha} D^\alpha u\,(s)\right)\right\|$$

and $t^\alpha D^\alpha S(t)$ is continuous on I because of Lemma 3 since S is differentiable. Selecting the different parameters in the coefficients in the relations (9) and (10) and the ones corresponding to $\left\| t^{\lambda_\alpha} D^\alpha u - t^{\lambda_\alpha} D^\alpha v \right\|$ sufficiently small, the contraction principle ensures the existence and uniqueness of a mild solution on I. □

4. Classical solutions

In this section we prove the existence and uniqueness of classical solutions to problem (1). We will use the short notation

$$\tilde{f}(t) = f\left(t, u(t), t^{\lambda_\alpha} D^\alpha u(t)\right),$$
$$\tilde{g}(t) = g\left(t, u(t), t^{\lambda_\nu} D^\nu u\,(t)\right).$$

Definition 4. A function $u(.) \in C^2(I; X)$ such that $t^{\lambda_\eta} D^\eta u \in C(I, X)$, $\eta = \alpha, \beta, \gamma, \nu$, is called a classical solution of (1) if $u(.) \in D(A)$ satisfies the equation in (1) and the initial conditions are verified.

Notice, from Lemma 3, that if $u(.) \in C^2(I; X)$ then $t^{\lambda_\eta} D^\eta u \in C(I; X)$, when $\lambda_\eta \ge \eta$.

The fact that $t^{\lambda_\alpha} D^\alpha u$ is Lipschitz on I will be used in a crucial manner. However, this is not always true. It turns out that this property holds when f is, in addition, Lipschitz in its first variable. This is what is proved next.

Proposition 1. *Assume that* (H1)–(H3) *hold,* $\lambda_\eta \ge \eta, \eta = \alpha, \beta, \gamma, \nu, u^0 + p(u, t^{\lambda_\beta} D^\beta u(t)) \in E \cap E_\eta$ *and* $u^1 + q(u, t^{\lambda_\gamma} D^\gamma u(t)) \in E_{\eta-1}$. *Let* $t^{\lambda_\alpha} D^\alpha C(t)u(0)$ *and* $t^{\lambda_\alpha} D^\alpha S(t)u'(0)$ *be Lipschitz on* I *and* u *be the mild solution of* (1). *If* f *is Lipschitz in its first variable, that is*

$$\|f(t, x, y) - f(s, x, y)\| \le B_f\, |t-s|, \quad t, s \in I,\ x, y \in X,$$

for some positive constant B_f, *then,* $t^{\lambda_\alpha} D^\alpha u$ *is Lipschitz on* I.

Proof. From (4), Lemmas 3 and 8 (recall that $0 < \alpha < 1$), we have

$$t^{\lambda_\alpha} D^\alpha u(t) = t^{\lambda_\alpha} D^\alpha C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] + t^{\lambda_\alpha} D^\alpha S(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right)\right]$$
$$+ t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s)\tilde{f}(s) ds + t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s) \int_0^s \tilde{g}(z) dz ds, \quad t \in I.$$

Therefore, for $t \in I$ and h such that $t + h \in I$, we can write

$$(t+h)^{\lambda_\alpha} D^\alpha u\,(t+h) - t^{\lambda_\alpha} D^\alpha u(t)$$
$$= \left((t+h)^{\lambda_\alpha} D^\alpha C(t+h) - t^{\lambda_\alpha} D^\alpha C(t)\right) u(0) + \left((t+h)^{\lambda_\alpha} D^\alpha S(t+h) - t^{\lambda_\alpha} D^\alpha S(t)\right) u'(0)$$
$$+ (t+h)^{\lambda_\alpha} \int_0^{t+h} I^{1-\alpha} C(t+h-s)\tilde{f}(s) ds - t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s)\tilde{f}(s) ds$$
$$+ (t+h)^{\lambda_\alpha} \int_0^{t+h} C(t+h-s) \int_0^s \tilde{g}\,(z)\, dz ds - t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s) \int_0^s \tilde{g}(z) dz ds$$

or by a change of variables

$$\begin{aligned}
&(t+h)^{\lambda_\alpha} D^\alpha u\,(t+h) - D^\alpha u\,(t)\\
&\quad= \left((t+h)^{\lambda_\alpha} D^\alpha C(t+h) - t^{\lambda_\alpha} D^\alpha C(t)\right) u(0) + \left((t+h)^{\lambda_\alpha} D^\alpha S(t+h) - t^{\lambda_\alpha} D^\alpha S(t)\right) u'(0)\\
&\qquad + (t+h)^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(s)\tilde{f}(t+h-s)\mathrm{d}s + (t+h)^{\lambda_\alpha} \int_t^{t+h} I^{1-\alpha} C(s)\tilde{f}(t+h-s)\mathrm{d}s\\
&\qquad - t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s)\tilde{f}(s)\mathrm{d}s + (t+h)^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(s) \int_0^{t+h-s} \tilde{g}(z)\mathrm{d}z\mathrm{d}s\\
&\qquad + (t+h)^{\lambda_\alpha} \int_t^{t+h} I^{1-\alpha} C(s) \int_0^{t+h-s} \tilde{g}(z)\mathrm{d}z\mathrm{d}s - t^{\lambda_\alpha} \int_0^t I^{1-\alpha} C(t-s) \int_0^s \tilde{g}(z)\mathrm{d}z\mathrm{d}s.
\end{aligned} \tag{11}$$

The relation (11) implies that there exist $K_i > 0$, $i = 1, 2$, such that

$$\begin{aligned}
&\left\|(t+h)^{\lambda_\alpha} D^\alpha u\,(t+h) - t^{\lambda_\alpha} D^\alpha u(t)\right\|\\
&\quad\le K_1 h + \frac{\tilde{M} T^{\lambda_\alpha+1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \left\|\tilde{f}(s+h) - \tilde{f}(s)\right\| \mathrm{d}s + T^{\lambda_\alpha} \int_0^h \left\|I^{1-\alpha} C(t+h-s)\tilde{f}(s)\right\| \mathrm{d}s\\
&\qquad + \frac{\tilde{M} T^{\lambda_\alpha+1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \left\|\int_s^{s+h} \tilde{g}(z)\mathrm{d}z\right\| \mathrm{d}s + T^{\lambda_\alpha} \int_0^h \left\|I^{1-\alpha} C(t+h-s) \int_0^s \tilde{g}(z)\right\| \mathrm{d}s\\
&\quad\le K_2 h + \frac{\tilde{M} T^{\lambda_\alpha+1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \left[B_f h + A_f \left(\|u(s+h) - u(s)\| + \left\|(s+h)^{\lambda_\alpha} D^\alpha u\,(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\|\right)\right] \mathrm{d}s.
\end{aligned}$$

This relation yields

$$\left\|(t+h)^{\lambda_\alpha} D^\alpha u\,(t+h) - t^{\lambda_\alpha} D^\alpha u(t)\right\| \le K_3 h + K_4 \int_0^t \left\|(s+h)^{\lambda_\alpha} D^\alpha u\,(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\| \mathrm{d}s$$

for some $K_3, K_4 > 0$ because u is Lipschitz on I, in fact u is differentiable since $u^0 + p(u, t^{\lambda_\beta} D^\beta u(t)) \in E$ and

$$\begin{aligned}
u'(t) &= AS(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] + C(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right)\right]\\
&\quad + \int_0^t C(t-s)\tilde{f}(s)\mathrm{d}s + \int_0^t C(t-s) \int_0^s \tilde{g}(z)\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

The Gronwall inequality allows us to conclude that $t^{\lambda_\alpha} D^\alpha u$ is Lipschitz on I. □

Lemma 10. *Assume that $\psi \in C^1([0, T])$ and $0 < \alpha < 1$; then*

$$\partial_h D^\alpha \psi(t) = D^\alpha \partial_h \psi(t) + \frac{\psi(0)\partial_h t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(-\alpha)} \frac{1}{h} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^h \frac{\psi(s) - \psi(0)}{(t+h-s)^{\alpha+1}} \mathrm{d}s$$

where $\partial_h v(t) := [v(t+h) - v(t)]/h$, $t \in (0, T]$ and h is such that $t + h \in (0, T]$.

Proof. This result is proved by using Definition 2 as follows:

$$\begin{aligned}
\partial_h D^\alpha \psi(t) &= \frac{1}{h}\left[D^\alpha \psi\,(t+h) - D^\alpha \psi(t)\right] = \frac{1}{h\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_0^{t+h} \frac{\psi(s)\mathrm{d}s}{(t+h-s)^\alpha} - \int_0^t \frac{\psi(s)\mathrm{d}s}{(t-s)^\alpha}\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} \partial_h \psi(t) + \frac{\mathrm{d}}{\mathrm{d}t} \partial_h \left(\frac{\psi(0) t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + \frac{1}{\Gamma(1-\alpha)} \frac{1}{h} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^h \frac{\psi(s) - \psi(0)}{(t+h-s)^\alpha} \mathrm{d}s\\
&= D^\alpha \partial_h \psi(t) + \frac{\psi(0)}{\Gamma(1-\alpha)} \partial_h \left(t^{-\alpha}\right) + \frac{1}{\Gamma(1-\alpha)} \frac{1}{h} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^h \frac{\psi\,(s) - \psi(0)}{(t+h-s)^\alpha} \mathrm{d}s, \quad t, t+h \in I. \quad \square
\end{aligned}$$

We are now ready to prove our result on the existence and uniqueness of a classical solution.

Theorem 2. *Suppose that* (H1)–(H3) *hold, $\lambda_\eta \ge \eta$, $\eta = \beta, \gamma, \nu$ and $\lambda_\alpha \ge \alpha + 1$. Assume further that $u(0) \in D(A)$ and $u'(0) \in E$ are such that $t^{\lambda_\alpha} D^\alpha C(t) u(0)$ and $t^{\lambda_\alpha} D^\alpha S(t) u'(0)$ are Lipschitz continuous on I. If f is continuously differentiable then the mild solution $u(t)$ of problem (1) is twice continuously differentiable and satisfies (1) on $[0, T]$ for some $T > 0$.*

Proof. As $u(0) \in D(A)$ and $u'(0) \in E$ it is clear from Lemma 3 that $u(0) \in E_\eta \cap E$ and $u'(0) \in E_{\eta-1}$. Therefore Theorem 1 ensures the existence and uniqueness of a mild solution u in $C_\eta^{RL}([0, T])$. In addition to that, since $u(0) \in E$, u is continuously differentiable and for $t \in I$

$$u'(t) = AS(t)u(0) + C(t)u'(0) + \int_0^t C(t-s)\tilde{f}(s)\mathrm{d}s + \int_0^t C(t-s)\int_0^s \tilde{g}(z)\mathrm{d}z\mathrm{d}s. \tag{12}$$

Let us consider the problem

$$\begin{aligned}\varphi(t) = {} & C(t)\left[Au(0) + \tilde{f}(0)\right] + AS(t)u'(0) + \int_0^t C(t-s)\left\{\tilde{f}_1(s) + \tilde{f}_2(s)u'(s)\right. \\ & \left. + \tilde{f}_3(s)\left[\lambda_\alpha s^{\lambda_\alpha - 1}D^\alpha u(s) + \frac{s^{\lambda_\alpha-(1+\alpha)}u(0)}{\Gamma(-\alpha)} + \frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)} + s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]\right\}\mathrm{d}s + \int_0^t C(t-s)\tilde{g}(s)\mathrm{d}s.\end{aligned} \tag{13}$$

Here $\tilde{f}_i$ denotes the partial derivative of f with respect to its i-th component. Clearly, all the terms in the right hand side of (13) are well-defined and (13) admits a unique solution $\varphi \in C([0,T])$ (see Theorem 1). We claim that $u'' = \varphi$ on I. To see this we will show that $\lim_{h\to 0}\left\|\partial_h u'(t) - \varphi(t)\right\| = 0$ where ∂_h and h are as in Lemma 10. First, (12) and (13) imply the relation

$$\begin{aligned}\partial_h u'(t) - \varphi(t) = {} & \partial_h AS(t)u(0) + \partial_h C(t)u'(0) + \partial_h\int_0^t C(t-s)\tilde{f}(s)\mathrm{d}s \\ & + \partial_h\int_0^t C(t-s)\int_0^s g\left(z, u(z), z^{\lambda_\nu}D^\nu u(z)\right)\mathrm{d}z\mathrm{d}s - C(t)\left[Au(0) + \tilde{f}(0)\right] \\ & - AS(t)u'(0) - \int_0^t C(t-s)\left\{\tilde{f}_1(s) + \tilde{f}_2(s)u'(s) + \tilde{f}_3(s)\right. \\ & \left. \times\left[\lambda_\alpha s^{\lambda_\alpha-1}D^\alpha u(s) + \frac{s^{\lambda_\alpha-(1+\alpha)}u(0)}{\Gamma(-\alpha)} + \frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)} + s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]\right\}\mathrm{d}s - \int_0^t C(t-s)\tilde{g}(s)\mathrm{d}s.\end{aligned} \tag{14}$$

We rearrange the terms in (14) side by side as follows:

$$\begin{aligned}\partial_h u'(t) - \varphi(t) = {} & \partial_h AS(t)u(0) - C(t)Au(0) + \partial_h C(t)u'(0) - AS(t)u'(0) \\ & + \frac{1}{h}\int_0^h C(t+h-s)\left[\tilde{f}(s) - \tilde{f}(0)\right]\mathrm{d}s + (\partial_h S(t) - C(t))\tilde{f}(0) \\ & + \int_0^t C(t-s)\left\{\partial_h\tilde{f}(s) - \tilde{f}_1(s) - \tilde{f}_2(s)u'(s)\right. \\ & \left. - \tilde{f}_3(s)\left[\lambda_\alpha s^{\lambda_\alpha-1}D^\alpha u(s) + \frac{s^{\lambda_\alpha-(1+\alpha)}u(0)}{\Gamma(-\alpha)} + \frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)} + s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]\right\}\mathrm{d}s \\ & + \partial_h\int_0^t C(t-s)\int_0^s \tilde{g}(z)\mathrm{d}z\mathrm{d}s - \int_0^t C(t-s)\tilde{g}(s)\mathrm{d}s.\end{aligned} \tag{15}$$

Notice that we have managed to obtain some nice terms by adding and subtracting $\partial_h S(t)\tilde{f}(0)$ and $\frac{1}{h}\int_0^t C(t+h-s)\tilde{f}(0)\mathrm{d}s$ which are actually equal.

For notational convenience we will denote by $l(h)$ a generic expression (which may change from line to line) satisfying $l(h) \to 0$ as $h \to 0$. As

$$\begin{aligned}\partial_h\int_0^t C(t-s)\int_0^s g\left(z, u(z), z^{\lambda_\nu}D^\nu u(z)\right)\mathrm{d}z\mathrm{d}s &= \partial_h\int_0^t C(s)\int_0^{t-s} g\left(z, u(z), z^{\lambda_\nu}D^\nu u(z)\right)\mathrm{d}z\mathrm{d}s \\ &= \int_0^t C(t-s)g\left(s, u(s), s^{\lambda_\nu}D^\nu u(s)\right)\mathrm{d}s + l(h)\end{aligned}$$

and by Proposition 1

$$\begin{aligned}\left\|\tilde{f}(s) - \tilde{f}(0)\right\| &\le A_f\left(\|u(s) - u(0)\| + \left\|s^{\lambda_\alpha}D^\alpha u(s) - \left(s^{\lambda_\alpha}D^\alpha u\right)(0)\right\|\right) + B_f\,|s| \\ &\le C_5 s\end{aligned}$$

the relation (15) (with the help of Lemma 1) yields

$$\begin{aligned}\left\|\partial_h u'(t) - \varphi(t)\right\| \le {} & l(h) + \left\|\int_0^t C(t-s)\left\{\partial_h\tilde{f}(s) - \tilde{f}_1(s) - \tilde{f}_2(s)u'(s)\right.\right. \\ & \left.\left. - \tilde{f}_3(s)\left[\lambda_\alpha s^{\lambda_\alpha-1}D^\alpha u(s) + \frac{s^{\lambda_\alpha-(1+\alpha)}u(0)}{\Gamma(-\alpha)} + \frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)} + s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]\right\}\mathrm{d}s\right\|.\end{aligned} \tag{16}$$

Furthermore, as f is continuously differentiable we may write

$$\begin{aligned}\tilde f(s+h)-\tilde f(s) &= \tilde f_1(s)h+\tilde f_2(s)\left(u(s+h)-u(s)\right)+\tilde f_3(s)\left((s+h)^{\lambda_\alpha}D^\alpha u(s+h)-s^{\lambda_\alpha}D^\alpha u(s)\right)\\ &\quad+\left\|\left(h,\,u(s+h)-u(s),\,(s+h)^{\lambda_\alpha}D^\alpha u(s+h)-s^{\lambda_\alpha}D^\alpha u(s)\right)\right\|_{I\times X^2}\\ &\quad\times\left\|\omega\left(\tilde f(s),\,h\partial_h u(s),\,h\partial_h\left(s^{\lambda_\alpha}D^\alpha u(s)\right)\right)\right\| \end{aligned}\tag{17}$$

where $\left\|\omega\left(\tilde f(s),\,h\partial_h u(s),\,h\partial_h\left(s^{\lambda_\alpha}D^\alpha u(s)\right)\right)\right\|\to 0$ when

$$\begin{aligned}&\left\|\left(h,\,u(s+h)-u(s),\,(s+h)^{\lambda_\alpha}D^\alpha u(s+h)-s^{\lambda_\alpha}D^\alpha u(s)\right)\right\|_{I\times X^2}\\ &\quad=|h|+\|u(s+h)-u(s)\|+\left\|(s+h)^{\lambda_\alpha}D^\alpha u(s+h)-s^{\lambda_\alpha}D^\alpha u(s)\right\|\to 0.\end{aligned}$$

The relations (16)–(17) and the identity

$$\partial_h\left(s^{\lambda_\alpha}D^\alpha u(s)\right)=D^\alpha u(s)\partial_h s^{\lambda_\alpha}+s^{\lambda_\alpha}\partial_h D^\alpha u(s)$$

imply that

$$\left\|\partial_h u'(t)-\varphi(t)\right\|\le l(h)+\left\|\int_0^t C(t-s)\tilde f_3(s)\left[s^{\lambda_\alpha}\partial_h D^\alpha u(s)-\frac{s^{\lambda_\alpha-(1+\alpha)}u(0)}{\Gamma(-\alpha)}-\frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)}-s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]\mathrm{d}s\right\|.$$

Taking into account Lemma 10, we get

$$\begin{aligned}\left\|\partial_h u'(t)-\varphi(t)\right\| &\le l(h)+\left\|\int_0^t C(t-s)\tilde f_3(s)\left\{s^{\lambda_\alpha}\left[\frac{u(0)}{\Gamma(1-\alpha)}\partial_h\left(s^{-\alpha}\right)-\frac{s^{-(1+\alpha)}u(0)}{\Gamma(-\alpha)}\right]\right.\right.\\ &\quad\left.\left.+\left[s^{\lambda_\alpha}D^\alpha\partial_h u(s)-s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]+\frac{s^{\lambda_\alpha}}{h\Gamma(-\alpha)}\int_0^h\frac{u(z)-u(0)}{(s+h-z)^{\alpha+1}}\mathrm{d}z-\frac{s^{\lambda_\alpha-\alpha}u'(0)}{\Gamma(1-\alpha)}\right\}\mathrm{d}s\right\|.\end{aligned}\tag{18}$$

Moreover, since

$$D^\alpha\partial_h u(t)=I^{1-\alpha}\partial_h u'(t)+t^{-\alpha}\frac{\partial_h u(0)}{\Gamma(1-\alpha)}$$

the inequality (18) can be rewritten as

$$\begin{aligned}\left\|\partial_h u'(t)-\varphi(t)\right\| &\le l(h)+\left\|\int_0^t C(t-s)\tilde f_3(s)\left\{s^{\lambda_\alpha}\left[\frac{u(0)}{\Gamma(1-\alpha)}\partial_h\left(s^{-\alpha}\right)-\frac{s^{-(1+\alpha)}u(0)}{\Gamma(-\alpha)}\right]\right.\right.\\ &\quad+\left[s^{\lambda_\alpha}I^{1-\alpha}\partial_h u'(s)-s^{\lambda_\alpha}I^{1-\alpha}\varphi(s)\right]+\frac{s^{\lambda_\alpha}}{h\Gamma(-\alpha)}\int_0^h\frac{u(z)-u(0)}{(s+h-z)^{\alpha+1}}\mathrm{d}z\\ &\quad\left.\left.+s^{\lambda_\alpha}\left[\frac{\partial_h u(0)s^{-\alpha}}{\Gamma(1-\alpha)}-\frac{s^{-\alpha}u'(0)}{\Gamma(1-\alpha)}\right]\right\}\mathrm{d}s\right\|.\end{aligned}$$

Clearly

$$\frac{s^{\lambda_\alpha}}{h\Gamma(-\alpha)}\int_0^h\frac{u(z)-u(0)}{(s+h-z)^{\alpha+1}}\mathrm{d}z\le\frac{\left|(s+h)^{-\alpha}-s^{-\alpha}\right|s^{\lambda_\alpha}}{h\Gamma(-\alpha+1)}\sup_{0\le z\le h}\|u(z)-u(0)\|.$$

If $s\le h$, then

$$\frac{|(s+h)^\alpha-s^\alpha|\,s^{\lambda_\alpha}}{hs^\alpha(s+h)^\alpha}\le(1+2\alpha)s^{\lambda_\alpha-\alpha-1}$$

and if $h\le s$, then

$$\frac{|(s+h)^\alpha-s^\alpha|\,s^{\lambda_\alpha}}{hs^\alpha(s+h)^\alpha}\le\frac{s^{\lambda_\alpha}}{h}\frac{\left(1+\frac{h}{s}\right)^\alpha-1}{(s+h)^\alpha}\le\alpha s^{\lambda_\alpha-\alpha-1}$$

where we have used the inequality $(1+x)^\delta\le 1+\delta x$. Therefore, by the continuity of u it is easy to see that

$$\begin{aligned}\left\|\partial_h u'(t)-\varphi(t)\right\| &\le l(h)+\left\|\int_0^t C(t-s)\tilde f_3(s)s^{\lambda_\alpha}\left[I^{1-\alpha}\partial_h u'(s)-I^{1-\alpha}\varphi(s)\right]\mathrm{d}s\right\|\\ &\le l(h)+L\int_0^t\sup_{0\le\sigma\le s}\left\|\partial_h u'(\sigma)-\varphi(\sigma)\right\|\mathrm{d}s\end{aligned}$$

where L is a positive constant (depending on T). By the Gronwall inequality we deduce that

$$\lim_{h\to 0}\left\|\partial_h u'(t)-\varphi(t)\right\|=0.$$

This, with Proposition 2.4 in [15], implies that $u(t)$ is a classical solution. The proof is complete. □

Examples. There exist several problems in applications which may fit into our problem (1). We can cite the Szabo model which arises in the effect of attenuation of for instance ultrasound second-harmonic imaging and high-intensity focused ultrasound beams for therapeutic surgery, the telegraph model which appears in the Brownian motion field, the Lokshin–Webster model which appears in the study of wave propagation and many others which appear in viscoelasticity, poroelasticity and porous media (see [26–29,25,23,24]). Simple models are of the form

$$\Delta p=\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}+\frac{2\alpha_0}{c_0}f(\alpha)D^\alpha p,$$

$$[D^2+aD^{1+\alpha}+bD]u-\operatorname{div}[A(x)\operatorname{grad}u]=f(x,t)$$

and

$$\frac{\partial^2 v}{\partial x^2}-A\frac{\partial^2 v}{\partial t^2}-B\int_0^t\frac{\partial^2 v/\partial\tau^2}{\sqrt{t-\tau}}\mathrm{d}\tau-C\frac{\partial v}{\partial t}=g(x,t)$$

augmented by nonlocal conditions.

Therefore an important class of problems in applications may be written in the following way:

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}u(t,x)=u_{xx}(t,x)+F\left(t,x,u(t,x),t^{\lambda_\alpha}D^\alpha u(t,x)\right)\\ \quad+\displaystyle\int_0^t G\left(t,x,u(t,x),t^{\lambda_\nu}D^\nu u(t,x)\right),\quad t\in I=[0,T],\ x\in[a,b]\\ u(t,a)=u(t,b)=0,\quad t\in I\\ u(0,x)=u^0(x)+\displaystyle\int_0^T P\left(u(s),s^{\lambda_\beta}D^\beta u(s)\right)(x)\mathrm{d}s,\quad x\in[a,b]\\ u'(0)=u^1(x)+\displaystyle\int_0^T Q\left(u(s),s^{\lambda_\gamma}D^\gamma u(s)\right)(x)\mathrm{d}s,\quad x\in[a,b]\end{cases}\tag{19}$$

in the space $X=L^2([0,\pi])$. We define the operator $Ay=y''$ with domain

$$D(A):=\{y\in H^2([0,\pi]):y(0)=y(\pi)=0\}.$$

This operator A has a discrete spectrum with $-n^2$, $n=1,2,\ldots$, as eigenvalues and $z_n(s)=\sqrt{2/\pi}\sin(ns)$, $n=1,2,\ldots$, as their corresponding normalized eigenvectors. Thus we may write

$$Ay=-\sum_{n=1}^{\infty}n^2(y,z_n)z_n,\quad y\in D(A).$$

Since $-A$ is positive and self-adjoint in $L^2([0,\pi])$, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$, which has the form

$$C(t)y=\sum_{n=1}^{\infty}\cos(nt)(y,z_n)z_n,\quad y\in X.$$

The associated sine family is found to be

$$C(t)y=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}(y,z_n)z_n,\quad y\in X.$$

One can also consider more general nonlocal conditions by allowing the Lebesgue measure ds to be of the form $\mathrm{d}\mu(s)$ and $\mathrm{d}\eta(s)$ for non-decreasing functions μ and η (or even more general: μ and η of bounded variation), that is

$$u(0,x)=u^0(x)+\int_0^T P\left(u(s),s^{\lambda_\beta}D^\beta u(s)\right)(x)\mathrm{d}\mu(s),$$

$$u_t(0,x)=u^1(x)+\int_0^T Q\left(u(s),s^{\lambda_\gamma}D^\gamma u(s)\right)(x)\mathrm{d}\eta(s).$$

These (continuous) nonlocal conditions cover, of course, the discrete cases

$$u\left(0,x\right)=u^{0}(x)+\sum_{i=1}^{n}\alpha_{i}u\left(t_{i},x\right)+\sum_{i=1}^{m}\beta_{i}t_{i}^{\lambda_{\beta}}D^{\beta}u\left(t_{i},x\right),$$
$$u_{t}\left(0,x\right)=u^{1}(x)+\sum_{i=1}^{r}\gamma_{i}u\left(t_{i},x\right)+\sum_{i=1}^{k}\lambda_{i}t_{i}^{\lambda_{\gamma}}D^{\gamma}u\left(t_{i},x\right)$$

which have been extensively studied by several authors in the integer order case.

For $u,v\in C([0,T];X)$ and $x\in[0,\pi]$, defining the operators

$$p(u,v)(x):=\int_{0}^{T}P\left(u(s),v(s)\right)(x)\mathrm{d}s,$$
$$q(u,v)(x):=\int_{0}^{T}Q\left(u(s),v(s)\right)(x)\mathrm{d}s,$$
$$g(t,u,v)(x):=G(t,x,u(t,x),v(t,x)),$$
$$f(t,u,v)(x):=F(t,x,u(t,x),v(t,x)),$$

allows us to write (19) abstractly in the form (1). Under appropriate conditions on F, G, P and Q which make the hypotheses hold for the corresponding f, g, p and q, our theorems ensure the existence of solutions to problem (19).

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